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On the reconstruction of Toeplitz matrix inverses from columns

Georg Heinig

Department of Mathematics and Computer Science, Kuwait University, P.O. Box 5969, 13060 Safat, Kuwait

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Abstract

In this paper we discuss the problem whether and how the inverse of a Toeplitz matrix can be recovered from some of its columns or parts of columns under the requirement that only $2n-1$ parameters are involved. The results generalize and strengthen earlier findings by Trench, Gohberg, Semencul, Krupnik, Ben-Artzi, Shalom, Labahn, Rodman and others. Special attention is paid to symmetric, skewsymmetric and hermitian Toeplitz matrix inverses and the question whether such a matrix can be retrieved from a single column. © 2002 Published by Elsevier Science Inc.

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1. Introduction

An $n \times n$ Toeplitz matrix $T = [a_{i-j}]_{i,j=0}^{n-1}$ depends on $2n-1$ parameters $a_{1-n}, \ldots, a_{n-1}$. Clearly, also its inverse, provided it exists, depends on these $2n-1$ parameters and the questions is where these parameters are hidden and how the whole inverse matrix can be recovered from them. In this paper we discuss the question to what extend the matrix can be recovered from a few columns.

A Toeplitz matrix itself T can easily be retrieved from its first and last columns. It is remarkable that this is the only pair of columns containing the full information about the matrix. In the cases of a symmetric, skewsymmetric or hermitian Toeplitz

E-mail address: georg@mcs.sci.kuniv.edu.kw (G. Heinig).

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matrix the full information about the matrix is contained both in the first column and in the last column but not in any other column.

It was first observed by Trench [9] and rediscovered by Gohberg and Semencul [3] that T^{-1} can be reconstructed from its first and last columns provided that the first component of the first column does not vanish. Trench gave a recursive formula for it and Gohberg and Semencul a matrix representation. Gohberg and Krupnik [2] observed that T^{-1} can be recovered from its first and the second columns if the last component of the first column does not vanish. Since in both cases two elements of the two columns coincide, due to the persymmetry of Toeplitz matrices, we can say that these two columns depend on $2n-1$ parameters. That means that there is no redundancy.

Let us point out that it follows from the two results just mentioned that for inverses of Toeplitz matrices there might be, in contrast to Toeplitz matrices, several pairs of columns containing the full information about the matrix.

In [1] Ben-Artzi and Shalom showed that three columns, namely the first one and two properly chosen adjacent ones, are always sufficient to reconstruct T^{-1}. This is unsatisfactory, since the three columns depend on $3n-3$ parameters, which means that there is a lot of redundancy.

In the paper [6] Labahn and Shalom improved the Ben-Artzi and Shalom result by showing that T^{-1} can be recovered from the first and another properly chosen column of T^{-1}. However, for the reconstruction of T^{-1} the knowledge of the first (or in another version the last) column of the original matrix T is required, so that, nevertheless, $3n-1$ parameters are needed to retrieve T^{-1}.

The result of Labahn and Shalom was recently rediscovered and provided with a simpler proof by Ng et al. in [7]. The latter work was the motivation for the author to write the present paper.

The question is now whether the information about the original matrix in the Labahn and Shalom result is redundant. In other words: can the inverse always be recovered from the first and another column? The answer to this question is NO. Let us give an instructive example.

Example 1.1. Consider a Toeplitz matrix of the form

$$T=\begin{bmatrix}0 & T_-\\ T_+ & 0\end{bmatrix},$$

where T_+ is an $m\times m$ nonsingular lower triangular and T_- is an $l\times l$ nonsingular upper triangular Toeplitz matrix, and $m+l=n$. Then T is nonsingular and

$$T^{-1}=\begin{bmatrix}0 & T_+^{-1}\\ T_-^{-1} & 0\end{bmatrix},$$

where T_+^{-1} is lower triangular Toeplitz and T_-^{-1} is upper triangular Toeplitz.

One can see that the first and any other column together do not contain the full information about T^{-1}. The only pair of columns containing the full information are the lth and $(l+1)$th column. That means in the Labahn and Shalom result the knowledge of the first column of the original matrix cannot be waived.

This example gives rise to the following questions:

1. If the first column of T^{-1} is known, is it possible to locate $n-1$ entries of T^{-1} so that T^{-1} can be recovered?
2. Under which conditions T^{-1} can be retrieved from its first column and another column?
3. Is the complete information on T^{-1} always contained in two of its columns?

The main aim of the present paper is to discuss these problems. In particular, we give an affirmative answer to the first and third questions and a partial answer to the second question in Section 2.

Furthermore, in Section 3 we study the specifics of symmetric and skewsymmetric Toeplitz matrices, and in Section 4 the specifics of hermitian Toeplitz matrices. Let us mention an earlier result for these special cases. In [8] it is shown that the inverse of a symmetric or hermitian Toeplitz matrix can be reconstructed from its first and another properly chosen column. This is unsatisfactory, since a symmetric or hermitian $n \times n$ Toeplitz matrix depends on n parameters and a skewsymmetric one only on $n-1$ parameters. Therefore, in this result information is redundant. The question is whether the inverse can be retrieved from a single column. We show that this is the case for skewsymmetric Toeplitz matrices confirming a conjecture from [5]. For symmetric Toeplitz matrices, however, our reconstruction problem has exactly two solutions if the first component of the first column of the inverse is zero. That means one bit more information, we call it *character* of T, is required to recover T^{-1}. For hermitian matrices the additional information that is needed to retrieve T^{-1} from one column is a complex number from the unit circle.

This leaves the following question open: can the inverse of a symmetric or hermitian Toeplitz matrix always be recovered from n of its entries?

2. General Toeplitz matrices

To start with, let us adopt some notations. Throughout the paper, let $\mathbb{F}$ denote a (commutative) field. We consider vectors with components from $\mathbb{F}$ and matrices with entries from $\mathbb{F}$. We regard vectors $\mathbf{z} \in \mathbb{F}^n$ as functions defined on $\{0, \ldots, n-1\}$ and we extend this function, if necessary, by zero to any greater domain, in particular the set of all integers $\mathbb{Z}$. If $\mathbf{z} = (z_k)_{k\in\mathbb{Z}}$, then we denote, for $i \leqslant j$, $\mathbf{z}(i : j) = (z_i, \ldots, z_j)$. We set $\mathbf{z}(i) = \mathbf{z}(i : i)$.

The columns of T^{-1} will be denoted by $\mathbf{x}_k$ $(k = 0, \ldots, n-1)$. Sometimes $\mathbf{x}_k$ will be consider as a vector in $\mathbb{F}^{n+1}$, i.e. we identify $\mathbf{x}_k = \mathbf{x}_k(0 : n-1)$ with $\mathbf{x}_k(0 : n)$. Instead of $\mathbf{x}_0(0 : n)$ we write $\mathbf{x}$. Let us point out that $\mathbf{x}$ will always have the length $n+1$.

Let $\mathbf{g} = [-a_{i-n}]_{i=0}^{n-1}$, where $a_{-n} \in \mathbb{F}$ is arbitrary but fixed,

$$\mathbf{y}' = T^{-1}\mathbf{g} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}' \\ 1 \end{bmatrix}.$$

The pair $\{\mathbf{x}, \mathbf{y}\}$ is called the *canonical fundamental system* for T. Note that it is a basis of the nullspace of the $(n-1) \times (n+1)$ Toeplitz matrix $\widetilde{T} = [a_{i-j}]$, where $i = 1, \dots, n-1$ and $j = 0, \dots, n$.

The following is proved in [4].

Theorem 2.1. *The inverse of the Toeplitz matrix T can be recovered from $\mathbf{x} = (x_k)_{k=0}^n$ and $\mathbf{y} = (y_k)_{k=0}^n$ via*

$$T^{-1} = \begin{bmatrix} x_0 & & 0 \\ \vdots & \ddots & \\ x_{n-1} & \dots & x_0 \end{bmatrix} \begin{bmatrix} y_n & \dots & y_1 \\ & \ddots & \vdots \\ 0 & & y_n \end{bmatrix} - \begin{bmatrix} y_0 & & 0 \\ \vdots & \ddots & \\ y_{n-1} & \dots & y_0 \end{bmatrix} \begin{bmatrix} x_n & \dots & x_1 \\ & \ddots & \vdots \\ 0 & & x_n \end{bmatrix}. \tag{2.1}$$

A matrix defined by the right-hand sides of (2.1), for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}^{n+1}$, is called the *Toeplitz Bezoutian* of $\mathbf{x}$ and $\mathbf{y}$. We denote it by $B(\mathbf{x}, \mathbf{y})$. This matrix is nonsingular if and only if the polynomials with coefficient vectors $\mathbf{x}$ and $\mathbf{y}$ are coprime and their coefficients of t^n do not both vanish. It is well known that the inverse of a nonsingular Toeplitz Bezoutian is a Toeplitz matrix (see [4]).

Since $B(\mathbf{x}, \mathbf{y}) = B(\mathbf{x}, \mathbf{y} + c\mathbf{x})$ for any $c \in \mathbb{F}$, $\mathbf{y}$ can be replaced in (2.1) by $\mathbf{y} + c\mathbf{x}$. The choice of c depends on the choice of a_{-n} in the vector $\mathbf{g}$.

Throughout the paper, let $\delta(T)$ denotes the number of zeros at the end of $\mathbf{x}$, i.e. $\delta(T) = l$ means $x_{n-l} \neq 0$ and $x_k = 0$ for $k > n - l$. Note that the integer $\delta(T)$ can also be described in terms of ranks of submatrices of T or in terms of determinants.

Suppose that $\delta(T) = l$. Then we may choose a_{-n} and thus $\mathbf{y}$ such that $y_{n-l} = 0$. With this choice $\mathbf{y}$ is unique.

The lth and $(l+1)$th columns of T^{-1} are given by

$$\mathbf{x}_{l-1} = \begin{bmatrix} x_0 & & 0 \\ \vdots & \ddots & \\ & & x_0 \\ x_{n-l} & & \\ & \ddots & \vdots \\ 0 & & x_{n-l} \end{bmatrix} \begin{bmatrix} y_{n-l+1} \\ \vdots \\ y_n \end{bmatrix} \tag{2.2}$$

and

$$\mathbf{x}_l = \begin{bmatrix} 0 & \dots & 0 \\ x_0 & & \\ \vdots & \ddots & \\ & & x_0 \\ x_{n-l} & & \\ & \ddots & \vdots \\ 0 & & x_{n-l} \end{bmatrix} \begin{bmatrix} y_{n-l+1} \\ \vdots \\ y_n \end{bmatrix} - x_{n-l}\mathbf{y}. \tag{2.3}$$

We split the vector $\mathbf{y}$ into two parts $\mathbf{y}_0 = \mathbf{y}(0 : n-l-1)$ and $\mathbf{y}_1 = \mathbf{y}(n-l+1 : n)$. Remember that $y_{n-l} = 0$ and $y_n = 1$. Furthermore, we introduce $(n-l) \times l$ and $l \times l$ matrices

$$L = \begin{bmatrix} 0 & \dots & 0 \\ x_0 & & 0 \\ \vdots & \ddots & \\ & & x_0 \\ \vdots & & \vdots \\ x_{n-l-2} & \dots & x_{n-2l-1} \end{bmatrix}, \quad U = \begin{bmatrix} x_{n-l} & \dots & x_{n-2l+1} \\ & \ddots & \vdots \\ 0 & & x_{n-l} \end{bmatrix},$$

Then (2.3) implies

$$L\mathbf{y}_1 - x_{n-l}\mathbf{y}_0 = \mathbf{x}_l(0 : n-l-1). \tag{2.4}$$

Now (2.2) and (2.3) lead to the following.

Lemma 2.1. *The vector* $\mathbf{y}_1$ *can be recovered from* $\mathbf{x}_0$ *and* $\mathbf{x}_{l-1}(n-l : n-1)$ *via*

$$\mathbf{y}_1 = U^{-1}\mathbf{x}_{l-1}(n-l : n-1). \tag{2.5}$$

The vector $\mathbf{y}_0$ *can be recovered from* $\mathbf{x}_0$, $\mathbf{y}_1$ *and* $\mathbf{x}_l(0 : n-l-1)$ *via*

$$\mathbf{y}_0 = \frac{1}{x_{n-l}} \left(L\mathbf{y}_1 - \mathbf{x}_l(0 : n-l-1)\right). \tag{2.6}$$

This answers the first question in Section 1 as follows.

Theorem 2.2. *If* $\delta(T) = l$, *then* T^{-1} *can be recovered from* $\mathbf{x}_0$, $\mathbf{x}_{l-1}(n-l : n-2)$ *and* $\mathbf{x}_l(0 : n-l-1)$ *with the help of formulas* (2.5), (2.6), *and Theorem* 2.1.

Note that the last component of $\mathbf{x}_{l-1}$ is equal to x_{n-l}. Thus the number of parameters in Theorem 2.2 equals $2n-1$. That means that there is no redundancy.

We now discuss the second question raised in Section 1. It follows from Theorem 2.1 that T^{-1} cannot be recovered from $\mathbf{x}_0$ and any column $\mathbf{x}_k$ for $k = 1, \dots, l-1$. In fact, these columns do not contain, for example, any information about the parameter y_1. We ask now under which condition T^{-1} can be recovered from $\mathbf{x}_0$ and $\mathbf{x}_l$.

It is easily verified that (2.3), if considered as a linear system with unknown vectors $\mathbf{y}_0$ and $\mathbf{y}_1$, is equivalent to (2.4) and

$$U'\mathbf{y}_1 = \mathbf{x}_l(n-l:n-1), \tag{2.7}$$

where U' is the $l \times l$ matrix

$$U' = \begin{bmatrix} x_{n-l-1} & \cdots & x_{n-2l} \\ & \ddots & \vdots \\ 0 & & x_{n-l-1} \end{bmatrix}.$$

This leads to the following.

Theorem 2.3. *If* $\delta(T) = l > 1$, *then* T^{-1} *can be recovered from the first and* $(l+1)$th *column if and only if* $x_{n-l-1} \neq 0$. *If the latter is the case, then* T^{-1} *can be retrieved with the help of* (2.7), (2.6) *and Theorem* 2.1.

We illustrate the result of Theorem 2.3 by an example.

Example 2.1. We choose $\mathbf{x} = (t, 1, 0, 0)$, $\mathbf{y} = (a, 0, b, 1)$. Then

$$B(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} t & 0 & 0 \\ 1 & t & 0 \\ 0 & 1 & t \end{bmatrix} \begin{bmatrix} 1 & b & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & 0 & a \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} t & bt & -a \\ 1 & t+b & bt \\ 0 & 1 & t \end{bmatrix}.$$

This matrix is nonsingular if $a + bt^2 - t^3 \neq 0$. It can be seen that in case that $t \neq 0$ the matrix can be reconstructed from the first and third columns. In case that $t = 0$ the knowledge of the second component of the second column is still necessary.

Theorem 2.3 does not completely answer the second question. One might conjecture that in case that $x_{n-l-1} = 0$ but $x_{n-l-2} \neq 0$, then T^{-1} can be recovered from $\mathbf{x}_0$ and $\mathbf{x}_{l+1}$. Some examples seem to support this conjecture. But in general this is not the case, as the following example shows.

Example 2.2. Let $\mathbf{x} = (0, 1, 0, 1, 0, 0)$ and $\mathbf{y} = (a, b, c, 0, d, 1)$. For almost all values of a, b, c, d the matrix $B(\mathbf{x}, \mathbf{y})$ is nonsingular and therefore the inverse of a Toeplitz matrix. The third column is given by $\mathbf{x}_2 = (-a, -b, d - c, 1, 0)$. From this $\mathbf{y}$ cannot be reconstructed. This confirms Theorem 2.3. The fourth column is given by $\mathbf{x}_3 = (0, c - a, -b, d, 1)$ which also does not contain the full information about $\mathbf{y}$ and so disproves the conjecture. In general, no pair of columns containing the first one has the full information about T.

We now consider the third question raised in Section 1. Assume that the lth and $(l+1)$th columns $\mathbf{x}_{l-1}$ and $\mathbf{x}_l$ are given. We want to reconstruct $\mathbf{x}$ and $\mathbf{y}$. For this we mention first that x_{n-l} is equal to the last component of $\mathbf{x}_{l-1}$. Then we observe that (2.2) and (2.3) yield

$$\mathbf{y} = \frac{1}{x_{n-l}} \left(\begin{bmatrix} 0 \\ \mathbf{x}_{l-1} \end{bmatrix} - \begin{bmatrix} \mathbf{x}_l \\ 0 \end{bmatrix} \right). \tag{2.8}$$

Finally we conlude from (2.2)

$$\begin{bmatrix} y_n & \dots & y_l \\ & \ddots & \vdots \\ 0 & & y_n \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_{n-l} \end{bmatrix} = \mathbf{x}_{l-1}(l-1 : n-1). \tag{2.9}$$

In view of $y_n = 1$ this system has a unique solution $\mathbf{x}(0 : n-l)$. Since the other components of $\mathbf{x}$ are zero, we have the complete information about $\mathbf{x}$. We arrived at the following.

Theorem 2.4. *If $\delta(T) = l$, then T^{-1} can be recovered from its* lth *and* $(l+1)$th *columns with the help of the formulas* (2.8), (2.9) *and Theorem* 2.1.

Example 2.2 illustrates this theorem. The second column equals $\mathbf{x}_1 = (0, d, 1, d, 1)$, the third one $\mathbf{x}_2 = (-a, -b, d-c, 1, 0)$. From these two columns a, b, c, d can be recovered and so the whole matrix.

If we replace T by its transpose T^{t} and take into account the persymmetry of Toeplitz matrices we obtain the following analogues of Theorems 2.2–2.4.

Corollary 2.1. *If $\delta(T^{\mathrm{t}}) = l$, then*:
1. T^{-1} *can be recovered from the last column* $\mathbf{x}_{n-1}$, $\mathbf{x}_{n-l}(1 : l-1)$ *and* $\mathbf{x}_{n-l-1}(l : n-1)$.
2. T^{-1} *can be recovered from* $\mathbf{x}_{n-1}$ *and* $\mathbf{x}_{n-l-1}$ *if and only if* $\mathbf{x}_n(l) \neq 0$.
3. T^{-1} *can be recovered from* $\mathbf{x}_{n-l-1}$ *and* $\mathbf{x}_{n-l}$.

We discuss another version. For this we define $\nu(T)$ as the numbers of zeros at the beginning of the vector $\mathbf{x}$, i.e. $\nu(T) = l$ means that $x_0 = \dots = x_{l-1} = 0$ and $x_l \neq 0$.

If $\nu(T) > 0$, then it is easily checked that the backward shifted vector $\mathbf{z} = \mathbf{x}(1 : n)$ satisfies $T\mathbf{z} = c\,\mathbf{e}_{n-1}$, where $\mathbf{e}_{n-1}$ is the last vector of the standard basis in $\mathbb{F}^n$ and $c \in \mathbb{F}$. We have $c \neq 0$, since otherwise T would be singular. Hence $\mathbf{x}_{n-1} = (1/c)\,\mathbf{z}$. From this we conclude that

$$\nu(T) = \delta(T^{\mathrm{t}}),$$

provided that $\nu(T) > 0$. That means that in Corollary 2.1 "$\delta(T^{\mathrm{t}}) = l$" can be replaced by "$\nu(T) = l$". Let us illustrate this by Example 2.2. In this example we have $\nu(T) = 1$. The last column of the inverse equals $\mathbf{x}_4 = (-a, 0, -a, 0, 0)$ and the fourth column equals $\mathbf{x}_3 = (0, c-a, -b, d, 1)$, from which a, b, c, d and so T^{-1} can be recovered.

We show now how the original matrix T can be employed to get some information about $\mathbf{y}$. This will lead to a modified version of the results in [6,7]. Suppose that

$\delta(T) = l$, i.e. $x_k = 0$ for $k > n - l$. Knowing the first column of T^{-1} and the first row of T, we introduce numbers

$$b_k = a_{-k}x_0 + \cdots + a_{1-n}x_{n-k-1}$$

for $k = 0, \ldots, l - 1$. Clearly, $b_0 = 1$.

Applying T to the identity (2.2) we obtain the following.

Lemma 2.2. *The vector* $\mathbf{y}_1$ *can be computed from* b_k $(k = 0, \ldots, l - 1)$ *via*

$$\begin{bmatrix} b_0 & \ddots & b_{l-1} \\ & \ddots & \vdots \\ 0 & & b_0 \end{bmatrix} \mathbf{y}_1 = \mathbf{e}_{l-1}, \tag{2.10}$$

where $\mathbf{e}_{l-1}$ *is the last vector of the standard basis of* $\mathbb{F}^l$.

Now $\mathbf{x}_{l-1}$ can be computed via (2.2). We arrived at the following.

Corollary 2.2. *If* $\delta(T) = l$, *then* T^{-1} *can be recovered from its first column* $\mathbf{x}_0$, *the part* $\mathbf{x}_l(0 : n - l - 1)$ *of its* $(l + 1)$th *column and the last column of T.*

If $\nu(T) > 0$, then instead of the numbers b_k we may introduce numbers

$$c_k = a_{n-1}x_k + \cdots + a_{k+1}x_{n-1}$$

from the first columns of T and T^{-1}. Then $\mathbf{x}_{n-1} = \frac{1}{c_0}\mathbf{x}(1 : n)$. A relation that is analogous to (2.10) leads then to the result of [7].

Corollary 2.3. *Let* $\nu(T) = l$. *Then* T^{-1} *can be recovered from its first and* $(n - l)$th *columns and the first column of T.*

3. Symmetric and skewsymmetric Toeplitz matrices

In this section we discuss the problem whether the inverse of a symmetric or skewsymmetric Toeplitz matrix can be reconstructed from one properly chosen column. Throughout the section we assume that the characteristic of the field $\mathbb{F}$ is not equal to 2.

If the first component x_0 of the first column of T^{-1} is nonzero, then T^{-1} can be recovered from the first column by the Gohberg–Semencul formula. It remains to consider the case $x_0 = 0$. In the skewsymmetric case we have always $x_0 = 0$.

We introduce the $k \times k$ matrix of the counteridentity

$$J_k = \left.\begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}\right\} k.$$

A vector $\mathbf{u} \in \mathbb{F}^k$ is called *symmetric* if $\mathbf{u} = J_k\mathbf{u}$ and it is called *skewsymmetric* if $\mathbf{u} = -J_k\mathbf{u}$.

The following is proved in [5].

Lemma 3.1. *If T is nonsingular and symmetric, then the nullspace of $\widetilde{T} = [a_{i-j}]$, $i = 1, \ldots, n-1,\ j = 0, \ldots, n$, is spanned by a symmmetric and a skewsymmetric vector; if T is nonsingular and skewsymmetric, then the nullspace of $\widetilde{T}$ is spanned by two symmetric vectors.*

Using this lemma we obtain the following.

Lemma 3.2. *Let T be such that $x_0 = 0$. If T is symmetric, then $\mathbf{x}$ is symmetric and $\mathbf{y}$ can be chosen as skewsymmetric or $\mathbf{x}$ is skewsymmetric and $\mathbf{y}$ can be chosen as symmetric. If T is skewsymmetric, then $\mathbf{x}$ and $\mathbf{y}$ are always symmetric.*

Proof. If $x_0 = 0$, then $\mathbf{x}' = \mathbf{x}(1 : n-1)$ belongs to the one-dimensional nullspace of the $(n-1) \times (n-1)$ Toeplitz matrix $T_{n-1} = [a_{i-j}]_{i,j=0}^{n-2}$. Since $J_{n-1}T_{n-1}J_{n-1} = T_{n-1}^{\mathrm{t}} = \pm T_{n-1}$, the vector $J_{n-1}\mathbf{x}'$ also belongs to this nullspace, i.e. $J_{n-1}\mathbf{x}' = \pm c\,\mathbf{x}'$ for some $c \in \mathbb{F}$. But J_{n-1} has only the eigenvalues ± 1. Thus $\mathbf{x}' \in \mathbb{F}^{n-1}$ and, therefore $\mathbf{x} \in \mathbb{F}^{n+1}$ are symmetric or skewsymmetric. It remains to apply Lemma 3.1. $\square$

Lemma 3.2 tells us that if $x_0 = 0$, then in the symmetric case there are two possibilities: $\mathbf{x}$ can be symmetric or skewsymmetric. In the first case we say that the *character* equals 1, in the second case we say that the character equals -1. From Lemma 3.2 we also conclude that if $x_0 = 0$, then $\nu(T) = \delta(T)$. In contrast to it, in the skewsymmetric case $\mathbf{x}$ is always symmetric. This conclusion was already observed in [5].

In the symmetric case the (symmetric or skewsymmetric) vector $\mathbf{y}$ (with $y_n = 1$) is always unique, which is not the case in the skewsymmetric case, but $\mathbf{y}$ becomes unique if we add the assumption $y_l = y_{n-l} = 0$, where $l = \delta(T)$.

We consider the case of symmetric nonsingular Toeplitz matrix T. In this case we do not have $y_{n-l} = 0$, so (2.3) modifies to

$$\mathbf{x}_l = \begin{bmatrix} & \mathbf{0} & \\ x_l & & 0 \\ \vdots & \ddots & \\ & & x_l \\ x_{n-l} & & \\ & \ddots & \vdots \\ 0 & & x_{n-l} \end{bmatrix} \begin{bmatrix} y_{n-l} \\ \vdots \\ y_n \end{bmatrix} - x_{n-l}\mathbf{y}, \tag{3.1}$$

where $\mathbf{0}$ is the $l \times (l+1)$ zero matrix. From (3.1) we obtain

$$\mathbf{x}_l(0) = -x_{n-l}\, y_0 = x_l$$

and

$$\mathbf{y}(0:l-1) = \mp \frac{1}{\mathbf{x}_l(0)}\, \mathbf{x}_l(0:l-1). \tag{3.2}$$

For the $(l+1)$th component we have

$$\mathbf{x}_l(l) = x_l\, y_{n-l} - x_{n-l}\, y_l = \mp 2 x_l\, y_l.$$

Hence

$$y_l = \mp \frac{1}{2\mathbf{x}_l(0)}\, \mathbf{x}_l(l). \tag{3.3}$$

Next we observe that relation (3.1) can be written in the form

$$\begin{aligned}
\mathbf{x}_l &= \begin{bmatrix} & \mathbf{0} & \\ y_{n-l} & & 0 \\ \vdots & \ddots & \\ & & y_{n-l} \\ y_n & & \\ & \ddots & \vdots \\ 0 & & y_0 \end{bmatrix} \mathbf{x}(l:n-l) - x_{n-l}\, \mathbf{y} \\
&= \mp \begin{bmatrix} & \mathbf{0} & \\ y_l & & 0 \\ \vdots & \ddots & \\ & & y_l \\ y_0 & & \\ & \ddots & \vdots \\ 0 & & y_0 \end{bmatrix} \mathbf{x}(l:n-l) \mp x_l\, \mathbf{y}.
\end{aligned} \tag{3.4}$$

Let $P_k^{\pm}$ denote the projections $P_k^{\pm} = \frac{1}{2}(I_k \pm J_k)$. The range of P_k^{+} is the subspace of symmetric vectors and the kernel is the subspace of skewsymmetric vectors in $\mathbb{F}^k$, and the other way round for P_k^{-}.

If the character of T equals 1, then we apply P_{n+1}^{+} to both sides of (3.4), if the character equals -1, then we apply P_{n+1}^{-} to both sides of (3.4). We obtain

$$P_{n+1}^{\pm}\mathbf{x}_l = \begin{bmatrix} w_0 & & 0 \\ \vdots & \ddots & \\ & & w_0 \\ w_{2l} & & \\ & \ddots & \vdots \\ 0 & & w_{2l} \end{bmatrix} \mathbf{x}(l:n-l),$$

where $\mathbf{w} = \mp(w_k)_{k=0}^{2l} = P_{2l+1}(0, \ldots, 0, y_l, \ldots, y_0)$. Now $\mathbf{x}(l:n-l)$ and with it $\mathbf{x}$ can be obtained from

$$\begin{bmatrix} w_{2l} & \ldots & w_l \\ & \ddots & \vdots \\ 0 & & w_{2l} \end{bmatrix} \mathbf{x}(l:n-l) = (P_{n+1}^{\pm}\mathbf{x}_l)(n-2l:n). \tag{3.5}$$

Since $w_{2l} = \pm 1$, this system has a unique solution. Once $\mathbf{x}$ and $\mathbf{y}(0:l)$ are given, $\mathbf{y}$ can be obtained via (3.1).

In the same way we can proceed in the skewsymmetric case. In the skewsymmetric case (3.3) turns into "$0 = 0$", so that y_l cannot be recovered from it. But we could assume that in this case $y_l = 0$.

Theorem 3.1. *Let T be a nonsingular symmetric or skewsymmetric Toeplitz matrix with $x_0 = 0$ and $\delta(T) = l$. Then the inverse can be recovered from the* $(l+1)$th *column of T^{-1} and, in the symmetric case, the knowledge of the character of T. In the symmetric case this can be done with the help of the formulas* (3.1)–(3.3), *and* (3.5).

For the skewsymmetric case this theorem proves a conjecture stated and proved for $l = 1$ in [5].

The inverse of a symmetric Toeplitz matrix T^{-1} cannot be uniquely recovered from the column $\mathbf{x}_l$ alone if the character of the matrix is unknown. In this case we will have always two solutions. Let us give an example.

Example 3.1. Let $\mathbf{x} = (0, 1, 1, 0)$ and $\mathbf{y} = (-1, -c, c, 1)$ with $c \neq 1$. Then $B(\mathbf{x}, \mathbf{y})$ is nonsingular, $l = 1$, and the corresponding Toeplitz matrix has character 1. Furthermore, $\mathbf{x}_1 = (1, 2c, 1)$. If we take $\mathbf{x} = (0, 1, -1, 0)$ and $\mathbf{y} = (1, c, c, 1)$, then T has character -1, but we have again $\mathbf{x}_1 = (1, 2c, 1)$.

4. Hermitian Toeplitz matrices

We assume in this section that $\mathbb{F} = \mathbb{C}$, the field of complex numbers, and consider hermitian Toeplitz matrices, i.e. we assume that $a_{-k} = \overline{a_k}$, where the bar denotes the conjugate complex. A vector $\mathbf{u} \in \mathbb{C}^n$ will be called *hermitian* if $J_n\mathbf{u} = \alpha\,\overline{\mathbf{u}}$ for some $\alpha \in \mathbb{C}$ from the unit circle. The number α will be called *character* of $\mathbf{u}$. The following is proved in the same way as Lemma 3.1.

Lemma 4.1. *If T is hermitian, then the nullspace of $\widetilde{T} = [a_{i-j}]$, $i = 1, \ldots, n-1$, $j = 0, \ldots, n$, is spanned by two hermitian vectors.*

The following can be checked by direct verification.

Lemma 4.2. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{n+1}$ be hermitan with characters α and β, respectively. Then the Bezoutian $B(\mathbf{u}, \mathbf{v})$ is an hermitian matrix if and only if $\alpha\beta = -1$.*

Using the arguments from the proof of Lemma 3.2 we conclude from Lemmas 4.1 and 4.2 the following.

Lemma 4.3. *Let T be an hermitian nonsingular Toeplitz matrix such that $x_0 = 0$. Then $\mathbf{x}$ is hermitian and $\mathbf{y}$ can be chosen hermitian. Moreover, if α and β are the characters of $\mathbf{x}$ and $\mathbf{y}$, respectively, then $\alpha\beta = -1$.*

The character of the hermitian vector $\mathbf{x}$ will be also called *character of T*. This agrees with the definition of character in the symmetric case if $\mathbb{F}$ are the reals.

We discuss now the uniqueness of the vector $\mathbf{y}$. In general, $\mathbf{y}$ can be replaced by $\widetilde{\mathbf{y}} = \mathbf{y} + c\,\mathbf{x}$ for any $c \in \mathbb{C}$, since $B(\mathbf{x}, \mathbf{y}) = B(\mathbf{x}, \mathbf{y} + c\,\mathbf{x})$. But we want $\widetilde{\mathbf{y}}$ to be hermitian with character $-\overline{\alpha}$. It can easily be checked that $\widetilde{\mathbf{y}}$ has this property if and only if $\overline{\alpha c} = -\alpha c$, that means if and only if $c = r\overline{\alpha}\sqrt{-1}$ for some real r.

Lemma 4.4. *Let T be an hermitian nonsingular Toeplitz matrix such that $x_0 = 0$ and $\delta(T) = l$. Then the vector $\mathbf{y}$ can be chosen such that $x_{n-l}y_l$ is real.*

Proof. Let α be the character of T. Then $\mathbf{y}$ can be chosen as an hermitian vector with character $-\overline{\alpha}$. Furthermore, $\mathbf{y}$ can be replaced by $\widetilde{\mathbf{y}} = \mathbf{y} - r\overline{\alpha}\,\mathbf{x}\sqrt{-1}$, where r is any real number. Let $\widetilde{\mathbf{y}} = (\widetilde{y}_k)_0^n$. We have

$$x_{n-l}\widetilde{y}_l = x_{n-l}y_l - r|x_l|^2\sqrt{-1}.$$

Since $x_l \neq 0$, we can choose

$$r = \operatorname{Im}\frac{x_{n-l}y_l}{|x_l|^2}.$$

In this case $x_{n-l}\widetilde{y}_l$ is real. □

We now can state the main result of this section.

Theorem 4.1. *Let T be a nonsingular hermitian Toeplitz matrix with $x_0 = 0$ and $\delta(T) = l$. Then the inverse can be recovered from the $(l+1)$th column of T^{-1} and the knowledge of the character of T.*

Proof. The proof follows the same lines as that of Theorem 3.1. We start with relation (3.1), which allows us first to evaluate $\mathbf{y}(0:l-1)$ via

$$\mathbf{y}(0:l-1) = -\frac{1}{\alpha \mathbf{x}_l(0)}\mathbf{x}_l(0:l-1),$$

in view of $\mathbf{x}_l(0) = \overline{x_l} = \overline{\alpha} x_{n-l}$.

Next we look at the $(l+1)$th component. We have $\mathbf{x}_l(l) = -2x_{n-l}y_l$. According to Lemma 4.4 we can choose $\mathbf{y}$ such that this number is real. We obtain

$$y_l = -\frac{\mathbf{x}_l(l)}{2\alpha \mathbf{x}_l(0)}.$$

Next we observe that relation (3.1) can be written in the form

$$\mathbf{x}_l = -\alpha \begin{bmatrix} & \mathbf{0} & \\ y_l & & 0 \\ \vdots & \ddots & \\ & & y_l \\ y_0 & & \\ & \ddots & \vdots \\ 0 & & y_0 \end{bmatrix} \mathbf{x}(l:n-l) - \alpha x_l\, \mathbf{y}.$$

We apply to both sides of this relation the (nonlinear) operator P_α defined by $P_\alpha \mathbf{u} = \frac{1}{2}\,(J_{n+1}\mathbf{u} + \overline{\alpha \mathbf{u}})$. Since $P_\alpha \mathbf{y} = 0$, we obtain

$$P_\alpha \mathbf{x}_l = \begin{bmatrix} w_0 & & 0 \\ \vdots & \ddots & \\ & & w_0 \\ w_{2l} & & \\ & \ddots & \vdots \\ 0 & & w_{2l} \end{bmatrix} \mathbf{x}(l:n-l),$$

where $\mathbf{w} = (w_k)_{k=0}^{2l} = P_\alpha(0,\dots,0,y_l,\dots,y_0)$. Now $\mathbf{x}(l:n-l)$, and with it $\mathbf{x}$, can be obtained from

$$\begin{bmatrix} w_{2l} & \dots & w_l \\ & \ddots & \vdots \\ 0 & & w_{2l} \end{bmatrix} \mathbf{x}(l:n-l) = (P_\alpha \mathbf{x}_l)(n-2l:n).$$

Since $w_{2l} \neq 0$, this system has a unique solution. Once $\mathbf{x}$ and $\mathbf{y}(0:l)$ are given, $\mathbf{y}(l+1:n)$ can be obtained via (3.1). $\square$

Example 4.1. Let $\mathbf{x} = (0, 1, \alpha, 0)$ and $\mathbf{y} = (-\overline{\alpha}, -\overline{\alpha}c, c, 1)$, where $|\alpha| = 1$ and c is a real number, $c \neq \operatorname{Re}\alpha$. Then $B(\mathbf{x},\mathbf{y})$ is nonsingular $l = 1$ and the corresponding Toeplitz matrix has character α. It is easily checked that $\mathbf{x}_1 = (1, 2c, 1)$, which is independent of α. This means that T^{-1} cannot be recovered from $\mathbf{x}_1$ alone.

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